Published by Institute of Physics Publishing for SISSA

Received: *November 3, 2006*
Accepted: *November 21, 2006*
Published: *December 18, 2006*

Jain states in a matrix theory of the quantum Hall effect

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Abstract: The U(N) Maxwell-Chern-Simons matrix gauge theory is proposed as an extension of Susskind's noncommutative approach. The theory describes D0-branes, nonrelativistic particles with matrix coordinates and gauge symmetry, that realize a matrix generalization of the quantum Hall effect. Matrix ground states obtained by suitable projections of higher Landau levels are found to be in one-to-one correspondence with the expected Laughlin and Jain hierarchical states. The Jain composite-fermion construction follows by gauge invariance via the Gauss law constraint. In the limit of commuting, "normal" matrices the theory reduces to eigenvalue coordinates that describe realistic electrons with Calogero interaction. The Maxwell-Chern-Simons matrix theory improves earlier noncommutative approaches and could provide another effective theory of the fractional Hall effect.

Keywords: Chern-Simons Theories, Matrix Models, Non-Commutative Geometry.

Contents

1. Introduction

In this paper we continue the study initiated in [1] of noncommutative and matrix gauge theories [2, 3] that could describe the fractional quantum Hall effect [4] as effective (non-relativistic) field theories.[1] In this approach, we aim at finding a consistent theoretical framework encompassing and completing the well-established phenomenological theories of Laughlin [5] and Jain [6], as well as the low-energy effective theories of edge excitations [7–10]. Our setting is formally rather different from that of non-relativistic fermions coupled to the Chern-Simons interaction developed by Fradkin and Lopez [11] and others [12], but there are some analogies in the physical interpretation.

The use of noncommutative and matrix theories was initiated by Susskind [13] who observed that two-dimensional semiclassical incompressible fluids in strong magnetic fields can be described by the non-commutative Chern-Simons theory in the limit of small noncommutative parameter θ, corresponding to high density. Afterwards, Polychronakos extended

[1] We refer to [1] for a more extensive introduction to noncommutative theories of the quantum Hall effect and the account of earlier literature.

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the theory to describe a finite droplet of fluid, and obtained the U(N) matrix gauge theory called Chern-Simons matrix model [14]. In this theory, the noncommutativity is realized in terms of two Hermitean matrix coordinates, $[X_1(t), X_2(t)] = i\,\theta\,\mathrm{I}_N$, and θ corresponds to a uniform background field.[2] Polychronakos analyzed the classical droplet solutions and its excitations and showed a very interesting relation with the one-dimensional Calogero model, whose particle have coordinates x_a, $a = 1, \ldots, N$, corresponding to the eigenvalues of one matrix, say X_1.

Several authors [15 − 22, 1] further analyzed the Chern-Simons matrix model trying to connect it to the physics of the fractional Hall effect, most notably the Laughlin states and its quasi-particle excitations. The allowed filling fractions of quantum fluids were found to be, $\nu = 1/(\mathbf{B}\theta + 1) = 1/(k+1)$, where $\mathbf{B}$ is the external magnetic field and $\mathbf{B}\theta$ is integer quantized by gauge invariance [23].

As in any gauge theory, the Gauss-law condition requires the states to be singlets of the U(N) gauge group. In matrix theories the singlet ground state wave function in eigenvalue coordinates x_a contains the Vandermonde factor, $\Delta(x) = \prod_{a<b}(x_a - x_b)$, that indicates the relation with one-dimensional fermions [24]. In the Chern-Simon matrix model, the contribution of the θ background forces the states to carry a specific representation of the U(N) gauge group: the non-trivial ground state wave function is found to be a power of the Vandermonde, $\prod_{a<b}(\lambda_a - \lambda_b)^{k+1}$, in terms of the complex eigenvalues λ_a, $a = 1, \ldots, N$, of $X = X_1 + i\,X_2$ [17]. This is actually the Laughlin wave function at filling $\nu = 1/(k+1)$ [5], upon interpreting the eigenvalues as the coordinates of N planar electrons in the lowest Landau level. Therefore, the celebrated Laughlin state was shown to be the exact ground state of the matrix theory that is completely determined by gauge invariance. This is the nicest result obtained in the matrix (noncommutative) approach.

In spite of these findings, the Chern-Simons matrix model so far presented some difficulties that limited its applicability as a theory of the fractional Hall effect [20]:

- The Chern-Simons matrix model does not possess quasi-particle excitations, only quasi-holes can be realized [14].
- The Jain states with the filling fractions, $\nu = m/(mk+1)$, $m = 2, 3, \ldots$, cannot be realized in the theory, even including more boundary terms [15].
- Even if the Laughlin wave function is obtained, the measure of integration differs from that of electrons in the lowest Landau level, owing to the noncommutativity of matrices [18]. As shown in ref. [1], the ground state properties of the matrix theory and of the Laughlin state agree at long distances but differ microscopically.
- Owing to the inherent noncommutativity, it is also difficult to match matrix observables with electron quantities of the quantum Hall effect [19].

In this paper, we show that some of these problems can be overcome by upgrading the Chern-Simons model to the Maxwell-Chern-Simons matrix theory. This includes an

[2]Here, I_N denotes the traceless $N \times N$ pseudo-identity matrix defined in section 2.

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additional kinetic term quadratic in time derivatives and the potential $V = -g\mathrm{Tr}\,[X_1, X_2]^2$, parametrized by the positive coupling constant g. All the terms in the action are fixed by the gauge principle because they are obtained by dimensional reduction of the three-dimensional Maxwell-Chern-Simons theory. The matrix theory has been discussed in the literature of string theory as the low-energy effective theory of a stack of N D0-branes on certain higher-brane configurations [25]; in particular, D0-branes have been proposed as fundamental degrees of freedom in string theory [3].

In section two, we introduce the Maxwell-Chern-Simons matrix theory, quantize the Hamiltonian and discuss the Gauss-law constraint on physical states. The Hamiltonian contains a kinetic term, involving matrix coordinates, X_1, X_2, and conjugate momenta, Π_1, Π_2, and the potential V parametrized by g. The kinetic term realizes a matrix analogue of the Landau levels: it decomposes into N^2 copies of the Landau Hamiltonian for the "particles" whose coordinates are the matrix entries, X^1_{ab}, X^2_{ab}, $a, b = 1, \ldots, N$. The energy scale is set by the Landau-level gap $\mathbf{B}/m$. The theory has an interesting phase diagram in terms of the parameters, $\mathbf{B}/m$ for the kinetic energy and g for the potential, while $\mathbf{B}\theta = k \in \mathbb{Z}$ is fixed by the choice of density, i.e. of filling fraction.

In section three we discuss the theory for $g = 0$ and find the physical gauge-invariant states by solving the Gauss law. In the zero-energy sector, corresponding to the lowest Landau level, the theory reduces to the earlier Chern-Simons matrix model and exhibits the Laughlin ground states. General $E > 0$ gauge-invariant states are SU(N) singlets that resemble Slater determinants of N fermions: therefore, we can set up a pseudo-fermionic Fock space description of matrix states, but find higher degeneracies due to matrix non-commutativity. We introduce a series of projections in the theory that gradually reduce the number of available states: in the m-projected theory, $m = 2, 3, \ldots$, we only allow states belonging to the lowest m matrix Landau levels. Each of the projected theories possesses non-degenerate ground states corresponding to the uniform filling of allowed levels, that remarkably match the pattern envisaged by Jain in his phenomenological theory of "composite fermions" [6]. These states have the expected filling fraction $\nu = m/(mk + 1)$, with $k = \mathbf{B}\theta$, and admit quasi-particle excitations.

To summarize, the Maxwell-Chern-Simons matrix theory at $g = 0$ can be suitably truncated to display non-degenerate ground states that are matrix generalizations of Jain's composite-fermion wave functions in the fractional Hall effect. This is the main result of this paper.

In section four, we switch on the potential, $V = -g\ \mathrm{Tr}\,[X_1, X_2]^2$. For $g \to \infty$, the matrices are forced to commute among themselves, and thus can be simultaneously diagonalized by unitary transformation:[3] $X^i = U\ \Lambda^i U^\dagger$, $\Lambda^i = \mathrm{diag}(\lambda^i_a)$, $i = 1, 2$, $a = 1, \ldots, N$. Therefore, $N(N-1)/2$ physical, non-diagonal matrix components are projected out. The momenta Π_1, Π_2 remain noncommutative, but their non-diagonal parts are completely determined by the Gauss-law condition [27, 16]. Once put back into the Hamiltonian, these terms induce a two dimensional Calogero interaction among the remaining degrees of freedom λ^i_a. The complete reduction to eigenvalues allows them to be interpreted as

[3]The complex matrix $X = X_1 + iX_2$ commutes with its adjoint and is called "normal" [26].

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coordinates of N electrons: their Calogero interaction $O(k^2/r^2)$ is a legitimate replacement of the Coulomb potential $O(e^2/r)$, because the universal properties of incompressible Hall fluids are rather independent of the specific form of potential, for large values of $\mathbf{B}$ [5, 28, 6]. Therefore, for $g \to \infty$ the Maxwell-Chern-Simons matrix theory definitely describes the fractional quantum Hall effect.

We remark that the $g = \infty$ theory cannot easily be solved: as in the original problem, there are no small parameters to expand and the non-perturbative gap is due to particle interaction. Nevertheless, we point out that the $g = \infty$ theory can be understood by relating it to the solvable $g = 0$ theory, following the deformation of $g = 0$ states caused by the potential $V = -g \, \mathrm{Tr}\, [X_1, X_2]^2$.

In section five, we discuss the phase diagram of the Maxwell-Chern-Simons theory as a function of g and $\mathbf{B}/m$. We note that the non-degenerate matrix Jain states found at $g = 0$ are not eigenstates of the potential V and evolve into other unknown states for $g > 0$. Nevertheless, the $g = 0$ matrix states evaluated for diagonal $X, \overline{X}$ expressions (i.e. at $g = \infty$) become Slater determinants that are exactly equal to the (unprojected) Jain wave functions [6]: extensive numerical results [5, 28, 6, 29] indicate that these states are very close to the exact energy eigenstates for several short-range interactions including $1/r^2$.

This fact let us to conjecture that the matrix ground states found at $g = 0$, corresponding to the Laughlin and Jain series, remain non-degenerate for all g values: namely, that there are no phase changes in the theory for the density values characteristic of $g \sim 0$ non-degenerate states.

Although in this paper we cannot provide a proof of this conjecture, we find it rather compelling and worth describing in some detail. We remark that approximate methods can treat perturbatively or self-consistently the $\mathrm{Tr}\, [X_1, X_2]^2$ potential around the $g = 0$ ground states, owing to their non-degeneracy. Our approach here is similar in spirit to that of the Lopez-Fradkin theory [11], where degeneracy was removed by adding the Chern-Simons interaction. If the V potential can be efficiently treated, the Maxwell-Chern-Simons matrix theory could provide a new effective theory of the fractional Hall effect, where the Laughlin and Jain ground states naturally appear at $g = 0$, owing to matrix gauge invariance, and get modified as $g \to \infty$ while remaining in the same universality class.

2. U(N) Maxwell-Chern-Simons matrix gauge theory

We start by discussing the canonical analysis of the Maxwell-Chern-Simons matrix theory [27, 16], in presence of the uniform background θ and the "boundary" term of ref. [14]. The theory involves three time-dependent $N \times N$ Hermitean matrices, $X_i(t)$, $i = 1, 2$ and $A_0(t)$, and the auxiliary complex vector $\psi(t)$: it is defined by the action,

$$S = \int dt \; \mathrm{Tr} \left[\frac{m}{2} (D_t \, X_i)^2 \; + \; \frac{\mathbf{B}}{2} \varepsilon_{ij} \, X_i \, D_t \, X_j \; + \; \frac{g}{2} [X_1, X_2]^2 \; + \; \mathbf{B}\theta \, A_0 \right] \\ -i \int \psi^\dagger \, D_t \psi \; . \tag{2.1}$$

The form of the covariant derivatives is: $D_t X_i = \dot{X}_i - i\,[A_0, X_i]$ and $D_t\psi = \dot{\psi} - iA_0\psi$.

For $m = 0$, the theory reduces to the Chern-Simons matrix model [13], fully analyzed in ref. [14]: indeed, in this limit the potential $\mathrm{Tr}\,[X_1, X_2]^2$ becomes a constant for all states (see section 2.2). Hereafter we set $m = 1$ and measure dimensionful constants accordingly. Under U(N) gauge transformations: $X_i \;\to\; UX_iU^\dagger$, $A_0 \;\to\; U\,(A_0 - id/dt)\,U^\dagger$, and $\psi \;\to\; U\psi$, the action changes by a total derivative, such that invariance under large gauge transformations requires the quantization, $\mathbf{B}\theta = k \in \mathbb{Z}$, as in the case of the Chern-Simons model [23].

The canonical momenta are the following Hermitean matrices:

$$\Pi_i \equiv \frac{\delta S}{\delta \dot{X}_i^T} \;=\; D_t X_i - \frac{\mathbf{B}}{2}\varepsilon_{ij} X_j\,, \tag{2.2}$$

and $\chi = \delta S/\delta\dot{\psi} = -i\psi^\dagger$, After Legendre transformation on these variables, one finds the Hamiltonian:

$$H \;=\; \mathrm{Tr}\left[\frac{1}{2}\left(\Pi_i + \frac{\mathbf{B}}{2}\,\varepsilon_{ij}\;X_j\right)^2 \;-\; \frac{g}{2}\,[X_1, X_2]^2\right]\,. \tag{2.3}$$

The variation of S w.r.t. the non-dynamical field A_0 gives the Gauss-law constraint; its expression in term of coordinate and momenta reads:

$$G \;=\; 0\,, \qquad\qquad G \;=\; i\;[X_1, \Pi_1] \;+\; i\;[X_2, \Pi_2] \;-\; \mathbf{B}\theta\;\mathrm{I} \;+\; \psi\otimes\psi^\dagger\,, \tag{2.4}$$

where I is the identity matrix. By taking the trace of G, one fixes the norm of the auxiliary vector ψ,

$$\mathrm{Tr}\;G \;=\; 0 \quad\longrightarrow\; \|\psi\|^2 = \mathbf{B}\theta N = kN\;. \tag{2.5}$$

We note that the auxiliary vector has vanishing Hamiltonian and trivial dynamics, $\psi(t) = \psi(0) = \mathrm{const.}$: as in the Chern-Simons model, it is necessary to represent the Gauss law on finite-dimensional matrices that have traceless commutators [14]. In a gauge in which all ψ components vanish but one, the term $\mathbf{B}\theta\;\mathrm{I}\;-\psi\otimes\psi^\dagger$ in (2.4) is replaced by the traceless "identity", $\mathbf{B}\theta\;\mathrm{I}_N$, $\;\mathrm{I}_N\,=\,diag(1,\cdots,1,1-N)$. At the quantum level, the operator G generates U(N) gauge transformations of X_i and ψ, and requires the physical states to be U(N) singlets subjected to the additional condition (2.5) counting the number of ψ_a components.

2.1 Covariant quantization

We now quantize all the $2N^2$ matrix degrees of freedom X^i_{ab} and later impose the Gauss law as a differential condition on wave functions in the Schroedinger picture. The Hamiltonian (2.3) for $g = 0$ is quadratic and easily solvable: the sum over matrix indices decomposes into N^2 identical terms that are copies of the Hamiltonian of Landau levels [8]. To see this, introduce the matrix:

$$A = \frac{1}{2\ell}\,(X_1 + i\;X_2) \;+\; \frac{i\ell}{2}\,(\Pi_1 + i\;\Pi_2)\;, \tag{2.6}$$

and its adjoint $A^\dagger$, involving the constant $\ell = \sqrt{2/\mathbf{B}}$ called "magnetic length".

The quantum commutation relations following from (2.2) are,

$$\left[\left[X^i_{ab}, \Pi^j_{cd}\right]\right] \;=\; i\ \delta^{ij}\ \delta_{ad}\ \delta_{bc}\,, \qquad\qquad \left[\left[\psi^\dagger_a, \psi_b\right]\right] \;=\; \delta_{ab}\ ; \tag{2.7}$$

they are written using double brackets to distinguish them from classical matrix commutators. The canonical commutators imply the following relations of N^2 harmonic oscillators:

$$\left[\left[A_{ab}, A^\dagger_{cd}\right]\right] \;=\; \delta_{ad}\ \delta_{bc}\ \ , \qquad\qquad [[A_{ab}, A_{cd}]] \;=\; 0\ . \tag{2.8}$$

Note that $A^\dagger$ is the adjoint of A both as a matrix and a quantum operator. The Hamiltonian can be expressed in term of A and $A^\dagger$ as follows:

$$H \;=\; \mathbf{B}\ \mathrm{Tr}\left(A^\dagger\ A\right) \;+\; \frac{\mathbf{B}}{2}N^2 \;-\frac{g}{2}\mathrm{Tr}\,[X_1, X_2]^2\ . \tag{2.9}$$

In the term $\mathrm{Tr}(A^\dagger A) = \sum_{ab} A^\dagger_{ab} A_{ba}$ one recognizes N^2 copies of the Landau level Hamiltonian corresponding to N^2 two-dimensional "particles" with phase-space coordinates, $\{\Pi^i_{ab}, X^i_{ab}\}$, $a, b = 1, \ldots, N$, $i = 1, 2$.

The one-particle state are also characterized by another set of independent oscillators corresponding to angular momentum excitations that are degenerate in energy and thus occur within each Landau level. To find them, introduce the matrix,

$$B = \frac{1}{2\ell}\,(X_1 - i\ X_2) \;+\; \frac{i\ell}{2}\,(\Pi_1 - i\ \Pi_2)\ , \tag{2.10}$$

and its adjoint $B^\dagger$. They obey:

$$\left[\left[B_{ab}, B^\dagger_{cd}\right]\right] \;=\; \delta_{ad}\ \delta_{bc}\ \ , \qquad\qquad [[B_{ab}, B_{cd}]] \;=\; 0\,, \tag{2.11}$$

and commute with all the $A_{ab}, A^\dagger_{ab}$.

The total angular momentum of the N^2 "particles" can be written in the U(N) invariant form

$$J \;=\; \mathrm{Tr}\,(X_1\ \Pi_2 \;-\; X_2\ \Pi_1) \;=\; \mathrm{Tr}\left(B^\dagger B \;-\; A^\dagger A\right)\ . \tag{2.12}$$

Therefore, the B oscillators count the angular momentum excitations of the particles within each Landau level. In conclusion, the $g = 0$ theory exactly describes N^2 free particles in the Landau levels. In section three we shall discuss the effect of gauge symmetry that selects the subset of multi-particle states obeying the Gauss law $G = 0$ (2.4).

2.2 Projection to the lowest Landau level and Chern-Simons matrix model

For large values of the magnetic field $\mathbf{B}$, one often considers the reduction of the theory to the states in lowest Landau level that have vanishing energy (2.9), i.e. obey $A_{ab} = 0\ \ \forall a, b$. All higher levels can be projected out by imposing the constraints $A = A^\dagger = 0$, that can be written classically (cf (2.6)):

$$\Pi_2 = \frac{\mathbf{B}}{2}\ X_1\,, \qquad\qquad \Pi_1 = -\frac{\mathbf{B}}{2}\ X_2\,, \tag{2.13}$$

corresponding again to vanishing kinetic term in the Hamiltonian (2.3).

In this projection, two of the four phase space coordinates per particle are put to zero: if we choose them to be Π_1, Π_2, the remaining variables X_1, X_2 become canonically conjugate. This can also be seen from the action (2.1), because the kinetic term $m\,(D_i X)^2$ vanishes and one is left with the Chern-Simons term implying the identification of one coordinate with a momentum [30, 8].

Upon eliminating Π_1, Π_2, the Gauss law (2.4) becomes:

$$G = -i\mathbf{B}\ [X_1, X_2] - \mathbf{B}\theta + \psi \otimes \psi^\dagger\ ; \tag{2.14}$$

namely, it reduces to the noncommutativity condition of the Chern-Simon matrix model, with action [14],

$$S_{\text{CSMM}} = \int\ dt\ \text{Tr}\left[\frac{\mathbf{B}}{2}\varepsilon_{ij}\ X_i\ D_t\ X_j\ +\ \mathbf{B}\theta\ A_0\right] - i\ \int \psi^\dagger\ D_t\psi\ . \tag{2.15}$$

Moreover, the potential term in the Hamiltonian (2.3) becomes a constant on all physical states verifying $G = 0$: one finds, using the normalization (2.5),

$$-\ \text{Tr}\,[X_1, X_2]^2\ =\ \text{Tr}\left(\theta\ \text{I} - \frac{1}{\mathbf{B}}\psi \otimes \psi^\dagger\right)^2\ =\ \theta^2\ N(N-1)\ . \tag{2.16}$$

In conclusion, the Hamiltonian (2.3) reduces to a constant, i.e. it vanishes. This completes the proof that the Maxwell-Chern-Simons matrix theory projected to the lowest Landau level is equivalent to the previously studied Chern-Simons matrix model. As shown in ref. [14], this theory is the finite-dimensional regularization of the noncommutative Chern-Simons theory [13]. In particular, the Gauss law (2.14) can be rewritten,

$$[X_1, X_2]\ =\ i\ \theta\ \text{I}_N\,, \qquad \text{I}_N =\ diag\,(1, \cdots, 1, 1-N)\ , \tag{2.17}$$

in the gauge in which ψ has only one non-vanishing component (the N-th one). Equation (2.17) expresses the coordinate noncommutativity in the limit $N \to \infty$ [31].

3. Physical states at $g = 0$ and the Jain composite-fermion correspondence

In this section we are going to solve the Gauss law condition (2.4) and find the gauge invariant states. We first recall the form of physical states in the lowest Landau level, equal to those of the Chern-Simons matrix model already found in ref. [14, 17] (section 3.1), and then discuss the general physical states (sections 3.3,3.4). We introduce the complex matrices,

$$\begin{aligned} X &= X_1 + i\ X_2\,, & \overline{X} &= X_1 - i\ X_2\,, \\ \Pi &= \frac{1}{2}\,(\Pi_1 - i\ \Pi_2)\ , & \overline{\Pi} &= \frac{1}{2}\,(\Pi_1 + i\ \Pi_2)\ , \end{aligned} \tag{3.1}$$

and use the bar for denoting the Hermitean conjugate of classical matrices, keeping the dagger for the quantum adjoint. We set the magnetic length to one, i.e. $\mathbf{B} = 2$. The wave functions of the Maxwell-Chern-Simons theory take the form,

$$\Psi\ =\ e^{-\text{Tr}\left(\overline{X}X\right)/2 - \overline{\psi}\psi/2}\ \Phi(X, \overline{X}, \psi)\ . \tag{3.2}$$

For energy and angular momentum eigenstates, the function Φ in (3.2) is a polynomial in the matrices $X, \overline{X}$ and the auxiliary field ψ. The integration measure reads:

$$\langle\Psi_1|\Psi_2\rangle \;=\; \int \mathcal{D}X\mathcal{D}\overline{X}\ \mathcal{D}\psi\mathcal{D}\overline{\psi}\ e^{-\mathrm{Tr}\overline{X}X-\overline{\psi}\psi}\ \Phi_1^*(X,\overline{X},\psi)\ \Phi_2(X,\overline{X},\psi)\ . \tag{3.3}$$

The operators A_{ab}, B_{ab} and $\psi_a, \psi^\dagger_b$ characterizing the Hilbert space (cf. section 2.2) become differential operators acting on wave functions:

$$\begin{aligned} A_{ab} &= \frac{1}{2}X_{ab} + i\overline{\Pi}_{ab} = \frac{X_{ab}}{2} + \frac{\partial}{\partial\overline{X}_{ba}}\,, \qquad A^\dagger_{ab} = \frac{\overline{X}_{ab}}{2} - \frac{\partial}{\partial X_{ba}}\,,\\ B_{ab} &= \frac{\overline{X}_{ab}}{2} + \frac{\partial}{\partial X_{ba}}\,, \qquad B^\dagger_{ab} = \frac{X_{ab}}{2} - \frac{\partial}{\partial\overline{X}_{ba}}\,, \qquad \psi^\dagger_a = \frac{\partial}{\partial\psi_a}. \end{aligned} \tag{3.4}$$

Correspondingly, the Gauss law condition (2.4) becomes:

$$\begin{aligned} &G_{ab}\ \Psi_{\mathrm{phys}}(X,\psi) \;=\; 0\,,\\ &G_{ab} = \sum_c \left(X_{ac}\,\frac{\partial}{\partial X_{bc}} - X_{cb}\,\frac{\partial}{\partial X_{ca}} + \overline{X}_{ac}\,\frac{\partial}{\partial\overline{X}_{bc}} - \overline{X}_{cb}\,\frac{\partial}{\partial\overline{X}_{ca}} \right) - k\ \delta_{ab} + \psi_a\,\frac{\partial}{\partial\psi_b} \end{aligned} \tag{3.5}$$

This operator acting on wave functions performs an infinitesimal gauge transformation of its variables: $X, \overline{X}, \psi$. Note that the expression of G_{ab} in (3.5) was normal ordered for this to obey the U(N) algebra [14].

The action of the angular momentum (2.12) on the polynomial part of wave functions is,

$$\sum_{ab}\left(X_{ab}\,\frac{\partial}{\partial X_{ab}} - \overline{X}_{ab}\,\frac{\partial}{\partial\overline{X}_{ab}} \right)\ \Phi(X,\overline{X},\psi) \;=\; \mathcal{J}\ \Phi(X,\overline{X},\psi)\ . \tag{3.6}$$

The eigenvalue $\mathcal{J}$ is just the total number of X matrices occurring in Φ minus that of $\overline{X}$. For states with constant density,[4] the angular momentum measures the extension of the "droplet of fluid", such that we can associate a corresponding filling fraction ν by the formula,

$$\nu \;=\; \lim_{N\to\infty}\ \frac{N(N-1)}{2\mathcal{J}}\ . \tag{3.7}$$

In a physical system of finite size, one can control the density of the droplet, i.e. the angular momentum, by adding a confining potential V_C to the Hamiltonian:

$$H \;\to\; H \;+\; V_C \;=\; H \;+\; \omega\ \mathrm{Tr}\left(B^\dagger\ B\right)\ . \tag{3.8}$$

and tune its strength ω. This potential is diagonal on all states and becomes quadratic in the lowest Landau level, $V_C \to \omega\mathrm{Tr}(\overline{X}X)$. Typical values for ω will be of order $\mathbf{B}/N$, that do not destroy the Landau-level structure but give a small slope to each level.

[4]See refs. [14, 1] for the definition of the density in the matrix theory.

3.1 The Laughlin wave function

The multi-particle states in the lowest Landau level obey the conditions $A_{ab}\Psi = 0$, $\forall\ a, b$: using (3.4) and (3.2), they read,

$$A_{ab}\ \Psi\ =\ 0 \qquad \longrightarrow \qquad \frac{\partial}{\partial \overline{X}_{ab}}\ \Phi(X, \overline{X}, \psi)\ =\ 0\,, \qquad \forall\ a, b\ . \tag{3.9}$$

They imply that the polynomial Φ does not contain any $\overline{X}_{ab}$, i.e. is a analytic function of the X_{ab} variables, as in the ordinary Landau levels. The gauge invariant states should be given by polynomials $\Phi(X, \psi)$ that satisfy the Gauss law (2.4), (3.5): namely, they are U(N) singlets that contain Nk copies of the ψ vector (from $\mathrm{Tr}G = 0$ in (3.5)). These solutions were already found in ref. [17] for the equivalent Chern-Simons matrix model: let us recall the argument. One can form a polynomial of an arbitrary number of X_{ab} and saturate the indices with ψ^a's and the U(N) invariant N-component epsilon tensor. For $k = 1$, one obtain the states:

$$\Phi_{\{n_1,\dots,n_N\}}\,(X, \psi)\ =\ \varepsilon^{a_1\dots a_N}\ \left(X^{n_1}\psi\right)_{a_1}\cdots\left(X^{n_N}\psi\right)_{a_N}\,, \qquad 0 \le n_1 < n_2 < \cdots < n_N\,, \tag{3.10}$$

for any set of positive ordered integers $\{n_i\}$. The ground state in the confining potential $\mathrm{Tr}(XX^\dagger)$ is given by the closest packing $\{0, 1, \dots, N-1\}$ that has the lowest angular momentum, i.e. lowest degree in X. Solutions for $k \neq 1$ are obtained by multiplying k terms (3.10), leading to the expressions, $\Phi_{\{n^1_1,\dots,n^1_N\}\cdots\{n^k_1,\dots,n^k_N\}}$.

As shown in ref. [17], an equivalent basis for these states is given by:

$$\begin{aligned}\Phi\,(X, \psi) &= \sum_{\{m_k\}}\ \mathrm{Tr}\,(X^{m_1})\cdots\mathrm{Tr}\,(X^{m_k})\ \Phi_{k,\ gs}\,, \\ \Phi_{k,\ gs} &= \left[\varepsilon^{a_1\dots a_N}\ \psi_{a_1}\ \left(X\psi\right)_{a_2}\cdots\left(X^{N-1}\psi\right)_{a_N}\right]^k\,,\end{aligned} \tag{3.11}$$

which is factorized into the ground state $\Phi_{k,\ gs}$ and the "bosonic" powers of X, $\mathrm{Tr}\,(X^{m_i})$, with positive integers $\{m_1, \dots, m_k\}$ now unrestricted.

Let us now suppose that we diagonalize the complex matrix X by the similarity transformation:

$$\begin{aligned} X &= V^{-1}\ \Lambda\ V\,, \qquad\qquad \Lambda\ =\ diag(\lambda_1, \dots, \lambda_N)\,, \\ \psi &= V^{-1}\ \phi\,, \end{aligned} \tag{3.12}$$

where the λ_a's are complex numbers. The dependence on V and ϕ factorizes in the ground state wave function and the powers of eigenvalues make up the Vandermonde determinant $\Delta(\lambda) = \prod_{a<b}(\lambda_a - \lambda_b)$, as follows:

$$\begin{aligned}\Phi_{k,\ gs}\,(\Lambda, V, \psi) &= \left[\varepsilon^{a_1\dots a_N}\ \left(V^{-1}\phi\right)_{a_1}\ \left(V^{-1}\Lambda\phi\right)_{a_2}\cdots\left(V^{-1}\Lambda^{N-1}\phi\right)_{a_N}\right]^k \\ &= \left[(\det V)^{-1}\ \det\left(\lambda_a^{i-1}\ \phi_a\right)\right]^k \\ &= (\det V)^{-k}\ \prod_{1\le a<b\le N}(\lambda_a - \lambda_b)^k\ \left(\prod_c\ \phi_c\right)^k\,.\end{aligned} \tag{3.13}$$

The central piece is indeed the Laughlin wave function for the ground state of the Hall effect [5], with eigenvalues as electron coordinates [17, 18]. The value of the filling fraction is:

$$\nu \ = \ \frac{1}{k+1}\,, \tag{3.14}$$

and is renormalized from the classical value (3.7), $\nu = 1/\mathbf{B}\theta = 1/k$, because the wave functions acquire one extra factor of $\Delta(\lambda)$ from the integration measure (3.3) reduced to the eigenvalues [18]. Physical values of ν correspond to even k's and antisymmetric wave functions. The factorized dependence on V and ϕ in (3.13) is the same for all the states (3.11), because they do not occur in the power sums $\mathrm{Tr}\,(X^r) = \sum_a \lambda_a^r$. The ψ dependence can be integrated out, while the dynamics of the additional degrees of freedom described by V will be discussed later (see section four).

As anticipated in the introduction, eq. (3.13) is the most intriguing result obtained in the noncommutative approach and the Chern-Simons matrix model: that of deriving the Laughlin wave function from gauge invariance in a matrix theory. Furthermore, Susskind's semiclassical analysis [13] showed that, in the limit $\theta \to 0$, this matrix state describes an incompressible fluid in high magnetic fields, with density,

$$\rho_o \ = \ \frac{1}{2\pi\theta}\,, \tag{3.15}$$

in agreement with the earlier identification of the filling fraction.

Let us discuss the excitations of the matrix Laughlin states. Multiplying the wave function by polynomials of $\mathrm{Tr}(X^r)$ as in (3.11), we find states with $\Delta\mathcal{J} = r$. These are the basis of holomorphic excitations over the Laughlin state. For $r = O(1)$, their energy given by the boundary potential, $\Delta E = \omega\Delta\mathcal{J} = O(r\,\mathbf{B}/N)$ is very small: they are the degenerate edge excitations of the droplet of fluid described by conformal field theories [7−10].

More interesting hereafter are the analogues of the quasi-hole and quasi-particle excitations of the Laughlin state, that are gapful localized density deformations. The quasi-hole is realized by moving one electron from the interior of the Fermi surface to the edge, causing $\Delta\mathcal{J} = O(N)$ and thus a finite gap $\Delta E = O(\mathbf{B})$. Its realization in the matrix theory is for example given by the state $\Phi_{\{n_1,\dots,n_N\}}$ in eq. (3.10), with $\{n_1, n_2, \cdots, n_M\} = \{1, 2, \cdots, N\}$. On the other hand, the quasi-particle excitation cannot be realized in the Chern-Simons matrix model [14].

3.2 The Jain composite-fermion transformation

The result (3.15) can actually be interpreted in the language of the Jain composite-fermion transformation [6]. According to Jain, a system of electrons with inverse filling fraction parametrized by:

$$\frac{1}{\nu} = \frac{\mathbf{B}}{2\pi\rho_o} = \frac{1}{m} + k\,, \qquad m = 1, 2, 3, \cdots\,, \tag{3.16}$$

can be mapped into a system of weakly interacting "composite fermions" at effective filling ν^*,

$$\frac{1}{\nu} \ \to \ \frac{1}{\nu^*} \ = \ \frac{1}{m}, \tag{3.17}$$

by removing (or "attaching") k quantum units of flux per particle (k even). From (3.16), the remaining effective magnetic field felt by the composite fermions is:

$$\mathbf{B} \ \to \ \mathbf{B}^* \ = \ B - \Delta\mathbf{B}\,, \qquad \Delta\mathbf{B} = k\ 2\pi\rho_o\ . \tag{3.18}$$

The relation between excluded magnetic field $\Delta\mathbf{B}$ and density is the key point of Jain's argument. The Lopez-Fradkin theory of the fractional Hall effect [11] implements this relation as the equation of motion for the added Chern-Simons interaction, upon tuning its coupling constant to $\kappa = 1/k$, and taking the mean-field approximation $\rho = \rho_o$.

Here we would like to stress that the Chern-Simon matrix model provides another realization of the Jain composite-fermion transformation (3.17), (3.18) for $m = 1$. For $k = 0$, the matrix theory reduced to the eigenvalues λ_a is equivalent to a system of free fermions in the lowest Landau level, i.e. to $\nu^* = 1$ [18, 1, 26]. In the presence of the θ background, the noncommutativity of matrix coordinates (2.17) forces the electrons to acquire a finite area of order θ, by the uncertainty principle, leading to the (semiclassical) density $\rho_o = 1/2\pi\theta$ (3.15) [13]. Using this formula of the density and the quantization of $\mathbf{B}\theta$, we re-obtain the Jain relation (3.18),

$$\mathbf{B}\theta \ = k \ \in \ \mathbb{Z} \quad \to \quad \mathbf{B} \ = \ k\ 2\pi\rho_o\ . \tag{3.19}$$

Given that noncommutativity is expressed by the Gauss law of the matrix theory, we understand that the mechanism for realizing the Jain transformation is analogous to that of the Lopez-Fradkin theory, but it is expressed in terms of different variables.

The results of the Chern-Simons matrix theory were however limited, because the (matrix analogues of) Jain states for $m = 2, 3, \ldots$ could not be found. In the following, we shall find them in Maxwell-Chern-Simons matrix theory.

3.3 General gauge-invariant states and their degeneracy

Consider first the case $k = 1$. The states in the lowest Landau level, i.e. the polynomials $\Phi_{\{n_1,\ldots,n_N\}}(X,\psi)$ in eq. (3.10), can be represented graphically as "bushes", as shown in figure 1a. The matrices X_{ab} are depicted as oriented segments with indices at their ends and index summation amounts to joining segments into lines, as customary in gauge theories. The lines are the "stems" of the bush ending with one ψ_a, represented by an open dot, and the epsilon tensor is the N-vertex located at the root of the bush. Bushes have N stems of different lengths: $n_1 < n_2 < \cdots < n_N$. The position i_ℓ of one X on the ℓ-th stem, $1 \le i_\ell \le n_\ell$, is called the "height" of X on the stem.

The general solutions of the $k = 1$ Gauss law (2.4), (3.5) will be Φ polynomials involving both X and $\overline{X}$: given that they transform in the same way under the gauge group (cf. (3.5), the polynomials will again have the form of bushes whose stems are arbitrary words of X and $\overline{X}$. Angular momentum and energy eigenstates are linear combinations of bushes with given number $\mathcal{J} = N_X - N_{\overline{X}}$. From the commutation relations (2.8), (2.11), energy and momentum eigenstates can be easily obtained by applying the $A^\dagger_{ab}$ and $B^\dagger_{ab}$ operators (3.4) to the empty ground state $\Psi_o = \exp\left(-\mathrm{Tr}\overline{X}X/2 - \overline{\psi}\psi/2\right)$. Their energy $E = \mathbf{B}N_A$ and momentum $\mathcal{J} = N_B - N_A$ are expressed in terms of the number of $A^\dagger$ and $B^\dagger$ operators,

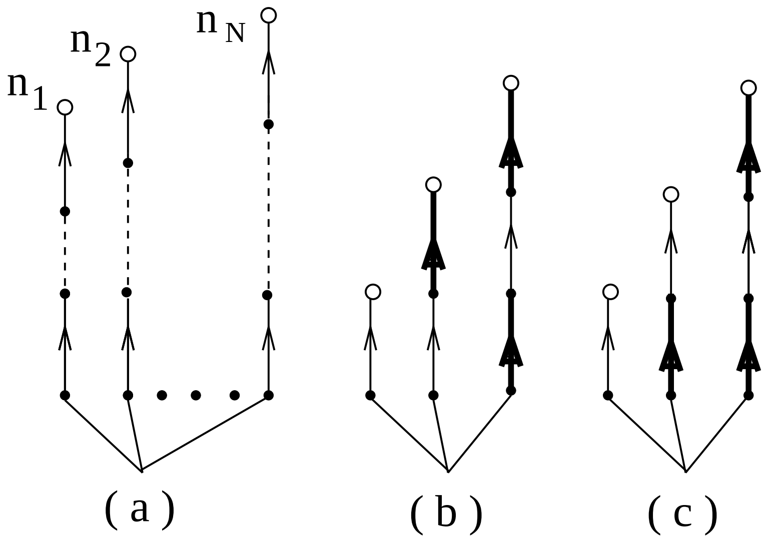

Figure 1: Graphical representation of gauge invariant states: (a) general states in the lowest Landau level (cf. eq. (3.10)); (b) and (c) examples in the second and third levels for $N = 3$.

N_A and N_B respectively. The polynomial part Φ of the wave function is thus expressed in the following variables:

$$\Psi \ = \ e^{-\mathrm{Tr}\ \overline{X}X/2-\overline{\psi}\psi/2}\ \Phi(\overline{B},\overline{A},\psi)\,, \qquad E=\mathbf{B}\ N_A\,, \quad J=N_B-N_A\,, \tag{3.20}$$

where $\overline{B} = X - \partial/\partial\overline{X}$ and $\overline{A} = \overline{X} - \partial/\partial X$ commute among themselves, $[[\overline{A}_{ab},\overline{B}_{cd}]] = 0$, and can be treated as c-number matrices. Since their U(N) transformations are the same as those of $X, \overline{X}$, they can be equivalently used to build the gauge invariant bush states. Examples of these general states are drawn in figure 1b, 1c, for $N = 3$: the variable $\overline{B}_{ab}$, replacing X_{ab} in the lowest Landau level, is represented by a thin segment, while $\overline{A}_{ab}$ is depicted in bold. Upon expanding $\overline{A}, \overline{B}$ in coordinates and derivatives acting inside Φ, one obtains in general a sum of $(X, \overline{X})$-bushes as anticipated.

The form of the general $k = 1$ gauge-invariant states suggests a pseudo-fermionic Fock-space representation involving N "gauge-invariant particles", as it follows:

- Each stem in the bush is considered as a "one-particle state" with quantum numbers, n_{Ai}, n_{Bi}, characterizing individual energies and momenta that are additive over the N particles, $N_A = \sum_{i=1}^{N} n_{Ai}$, $N_B = \sum_{i=1}^{N} n_{Bi}$.

- Since two stems cannot be equal, one should build a Fermi sea of N such one-particle states.

- The one-particle states form again Landau levels with energies $\varepsilon_i = \mathbf{B} n_{Ai}$, but there are additional degeneracies at fixed momentum with respect to the ordinary system; actually, in each stem, all possible words of $\overline{A}$ and $\overline{B}$ of given length yield independent states, owing to matrix noncommutativity (assuming large values of N).

Such "gauge invariant Landau levels" are shown in figure 2, together with their degeneracies, $(n_A+n_B)!/n_A!n_B!$, given by the number words of two letters with multiplicities n_A and n_B. These gauge invariant states should not be confused with the Landau levels discussed in section two, that are relative to the states of the N^2 gauge variant "particles" with $X_{ab}, \overline{X}_{ab}$ coordinates. The analysis of some examples shows that the gauge invariant states are many-body superpositions of the former N^2 states that are neither bosonic nor fermionic and thus rather difficult to picture. Instead, the interpretation in terms of N fermionic "gauge-invariant particles" is rather simple and also convenient for the physical limit $g=\infty$ of commuting matrices (to be discussed in section four). Finally, the gauge invariant states solution of the $k>1$ Gauss law are given by tensoring k copies of the structures just described, in complete analogy with the lowest-level solutions (3.11). Thus there are k Fermi seas to be filled with N "gauge-invariant particles" each.

In the following, we are going to introduce a set of projections of the $g=0$ Maxwell-Chern-Simons theory that will reduce the huge degeneracy of matrix states.

Degeneracies are better accounted for in a finite system, so we first modify the Hamiltonian to this effect. For example, the quadratic confining potential V_C (3.8) permits degenerate states that have equal energy and angular momentum — this also occurs in the ordinary Landau levels. The problem can be solved by using finite-box boundary conditions, that can be simulated by modifying the confining potential V_C in the Hamiltonian (3.8) as follows:

$$V_C \;=\; \omega \;\mathrm{Tr}\left(B^\dagger\; B\right) \;+\; \omega_n\;\mathrm{Tr}\left(B^{\dagger n}\; B^n\right)\,, \tag{3.21}$$

for a given value of n. The added operator $\mathrm{Tr}\left(B^{\dagger n}\; B^n\right)$ commutes with the $g=0$ Hamiltonian and angular momentum and has the following spectrum: when acting on stems, each B_{ba} is a derivative that erases one $\overline{B}_{ab}$ matrix and fixes the indices at the loose ends of the stem to a and b respectively. Next, further $(n-1)$ derivatives act, with index summations, and finally the length-n strand $\overline{B}^n_{ab}$ is added to complete a new bush without cut strands. On stems with $n_B \geq n$, this operator has a diagonal action with eigenvalue $O(N^{n-1})$; on other strands, it is non-diagonal with $O(1)$ coefficients. Therefore, in the limit of large N and in the physical regime $n_A \ll n_B$, the confining potential (3.21) effectively realizes the finite-box condition $n_{Bi} \leq n$ for all Landau levels.

In a finite system of size n, the degeneracy of the k-th "gauge invariant Landau level" is $O(n^k/k!)$ and the total degeneracy grows exponentially, $O(\exp(n))$, for large energy. In presence of a quadratic confining potential, it would grow exponentially with the energy and give rise to a Hagedorn transition at finite temperature. Here one rediscovers a known property of matrix theories that makes them more similar to string theories than to field theories of ordinary matter [3, 25].

In our physical setting, we should consider this feature as a pathology of the $g=0$ theory that should be cured in some way. Actually, for $g>0$ the potential term in the Hamiltonian, $V=(g/8)\mathrm{Tr}\left[X,\overline{X}\right]^2$, tends to eliminate the degeneracy due to matrix noncommutativity, as it follows. Consider a pair of degenerate states at $g=0$, that differ for one matrix commutation, such those shown in figure 1b,1c, and call their sum and

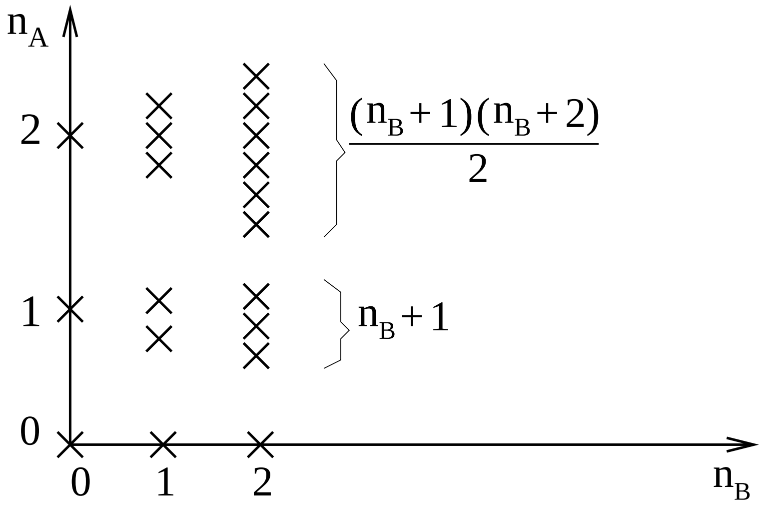

Figure 2: Pseudo-fermionic Fock space representation of gauge invariant states for $k = 1$.

difference Ψ_+ and Ψ_-, respectively. For large values of g, the state Ψ_+ can have a finite energy, while Ψ_- will acquire a growing energy, corresponding to the freezing of the degrees of freedom of the commutator.

Therefore, the Maxwell-Chern-Simons matrix theory for large values of g possesses degeneracies that are consistent with ordinary two-dimensional matter; indeed, in section four we shall show that the theory at $g = \infty$ reduces to the ordinary quantum Hall effect with $O(1/r^2)$ interparticle interactions. In conclusion, the matrix degeneracies at $g = 0$ can be dealt with by the theory itself by switching the V potential on.

This fact is however not particularly useful from the practical point of view, because we do not presently know how to compute the spectrum of the theory for $g > 0$. Precisely as in the original problem of the quantum Hall effect, the free theory is highly degenerate and the degeneracy is broken by the interaction (potential). The introduction of matrix variables would not appear as a great improvement towards solving this problem, given that their degeneracies are actually larger.

In spite of this, we shall find that matrix states do capture some features of the quantum Hall dynamics. In the following we shall introduce truncations of the $g = 0$ matrix theory that will eliminate most degeneracies and will naturally select non-degenerate ground states that are in one-to-one relation with the Jain hierarchical states [6].

3.4 The Jain ground states by projection

We reconsider the lowest Landau level condition (3.9) $A_{ab}\Psi = 0$, $\forall a, b$, that singles out the Laughlin wave functions as the unique ground states at filling fractions $\nu = 1/(k + 1)$. Although apparently not gauge invariant, it follows from the gauge invariant condition of vanishing energy, because the Hamiltonian, $H = 2\sum_{ab} A^\dagger_{ab} A_{ab}$, is a sum of positive operators that should all individually vanish.

Consider now the weaker condition, $(A_{ab})^2\,\Psi = 0$, $\forall\ a,b$, allowing the N^2 gauge variant "particles" to populate the first and second Landau levels. On the polynomial wave function, this projection reads:

$$\left(\frac{\partial}{\partial \overline{A}_{ab}}\right)^2 \Phi(\overline{A},\overline{B},\psi) \ = \ 0\,, \qquad \forall\ a,b\ . \tag{3.22}$$

The solutions are polynomials that are at most linear in each gauge-variant component $\overline{A}_{ab}$: one can think to an expansion of Φ in powers of $\overline{A}_{ab}$ that must stop at finite order as in the case of Grassmann variables. The condition $(A_{ab})^2\,\Psi = 0$ cannot be implemented as a gauge invariant term in the Hamiltonian, because the corresponding sum of positive operators, $\sum_{ab}\ A^{\dagger 2}_{ab}\ A^2_{ab}$, would not be gauge invariant. Nevertheless, when acting on the bush states described in the previous section (cf. figure 1), this condition respects gauge invariance.[5] In summary, this condition defines a gauge invariant truncation of the $g = 0$ matrix theory that cannot be easily realized by a change in the Hamiltonian.

In the following we shall study the truncated matrix theories that are defined by the projections: $(A_{ab})^m\,\Psi = 0$, for m taking the successive values $2, 3, 4, \ldots$; their wave functions contain the N^2 gauge variant "particles" filling the lowest m Landau levels.

We first discuss the theory with second level projection $A^2 = 0$: we outline the solutions of condition (3.22) leaving the details to appendix A. Let us try to insert one or more $\overline{A}$ at points on the bush and represent them as bold segments, as in figure 1. The differential operator (3.22) acts by sequentially erasing pairs of bold lines on the bush in any order, each time detaching two branches and leaving four free extrema with indices fixed to either a or b, with no summation on them. For example, when acting on a pair of $\overline{A}$ located on the same stem, it yields a non-vanishing result: this limits to one $\overline{A}$ per stem. Cancellations can occur for pairs of $\overline{A}$ on different stems, owing to the antisymmetry of the epsilon tensor, as it follows:

$$\left(A^b_a\right)^2 \Phi = \cdots\ +\ \varepsilon^{\cdots i \cdots j \cdots}\left(\cdots M_{ia}\ N_{ja} \cdots V^b\ W^b\right)\ +\ \cdots\,, \qquad (a,b \text{ fixed}), \tag{3.23}$$

that vanishes whenever $M = N$. The analysis of appendix A shows that there cannot be further cancellations involving linear combinations of different bushes. Therefore, the general solution of (3.22) is a bush involving one $\overline{A}$ per stem (max N matrices in total), all of them located at the same height on the stems, as follows:

$$\begin{aligned}\Phi^{(II)}_{\{n_1,\ldots,n_\ell;p;n_{\ell+1},\ldots,n_M\}} &= \varepsilon^{i_1\ldots i_N} \prod_{k=1}^{\ell}\left(\overline{B}^{n_k}\psi\right)_{i_k} \prod_{k=\ell+1}^{N}\left(\overline{B}^p\overline{A}\ \overline{B}^{n_k}\psi\right)_{i_k}\,, \\ & \qquad 0 \le n_1 < \cdots < n_\ell\,, \quad 0 \le n_{\ell+1} < \cdots < n_N\ .\end{aligned} \tag{3.24}$$

These wave functions can be related to many-body states of ordinary Landau levels: assuming diagonal expressions for both $\overline{B}$ and $\overline{A}$, the matrix states become Slater determinants of N electron one-particle states [6, 29]. This relation is surjective in general, because states differing by matrix orderings get identified; however, for states of form (3.24), the

[5] Instead, non-vanishing eigenstates of A^2_{ab} are gauge variant.

matrix degeneracy is limited to the p dependence. This shows how the projection $A^2 = 0$ works in reducing degeneracies.

Let us analyze the possible matrix states in the $A^2 = 0$ theory with finite-box conditions, referring to figures 1, 2 for examples. The most compact state corresponds to *homogeneous* filling all the allowed states in the first and second Landau levels with $N/2$ "gauge invariant particles" each; it reads:

$$\Phi_{1/2,\ gs} = \varepsilon^{i_1 \dots i_N} \prod_{k=1}^{N/2} \left(\overline{B}^{k-1} \psi \right)_{i_k} \prod_{k=1}^{N/2} \left(\overline{A}\ \overline{B}^{k-1} \psi \right)_{i_{N/2+k}} , \tag{3.25}$$

with angular momentum $\mathcal{J} = N(N-4)/4$. One easily sees that this state is non-degenerate for boundary conditions enforcing maximal packing, $n_{Bi} \leq N/2$, due to the vanishing of the p parameter in (3.24). Assuming homogeneity of its density and using (3.7), we can assign it the filling fraction $\nu^* = 2$.

Let us now discuss the states in the $A^2 = 0$ theory for generic k values. Gauge invariant states should be products of k bushes, as in (3.11): they survive the projection (3.22), provided that the two derivatives always vanish when distributed over all bushes. Given the product state with one bush of type (3.25), obeying $A^2\ \Phi_{1/2,\ gs} = 0$,

$$\Phi_{k+1/2,\ gs} \ = \ \Phi_{k-1,\ gs}\ \Phi_{1/2,\ gs}\, , \tag{3.26}$$

the other factor involving $k-1$ bushes should satisfy $A\ \Phi_{k-1,\ gs} = 0$ and actually be the Laughlin state (3.10). The state (3.26) is also non-degenerate with appropriate tuning of the boundary potential. From the $\mathcal{J}$ value, one can assign the filling fraction,[6] $1/\nu = k + 1/2$, to this state.

We thus find the important result that the projected Maxwell-Chern-Simons theory possesses non-degenerate ground states that are the matrix analogues of the Jain states obtained by composite-fermion transformation at $\nu^* = 2$, eqs. (3.16), (3.17). The matrix states (3.26), (3.25) would actually be equal to Jain's wave functions, if the $\overline{A}, \overline{B}$ matrices were diagonal: the ψ dependence would factorize and the matrix states reduce to the Slater determinants occurring in Jain's wave functions (before the projection to the lowest Landau level). Indeed, the diagonal limit can be obtained as follows. We note that the derivatives present in the expressions (3.20) of $\overline{A}$ and $\overline{B}$ vanish when acting on the states (3.25) due to antisymmetry of the epsilon tensor:[7] in the expressions of these states we can replace, $\overline{B} \to X, \overline{A} \to \overline{X}$. Therefore, the Jain and matrix states become identical in the limit of diagonal $X, \overline{X}$, that is realized for $g \to \infty$ as discussed in section four.

The correspondence extends to the whole Jain series: the other $\nu^* = m$ non-degenerate ground states are respectively obtained in the theories with $A^m = 0$ projections. Before discussing the generalization, let us analyze the other allowed states by the $A^2 = 0$ projection. They are obtained by relaxing the boundary conditions for (3.25), i.e. by reducing

[6]Keeping in mind the contribution of 1 from the Vandermonde of the integration measure.

[7]Using the graphical rules introduced before (cf. figure 1), this simplification is found for all states where Landau levels are homogeneously (i.e. completely) filled with non-increasing number of particles, namely $N_1 \geq N_2 \geq \cdots \geq N_m$, with $\sum_{i=1}^{m} N_i = N$.

the density of the system, allowing for lower fillings of the "gauge invariant Fermi sea". The non-degenerate Laughlin ground state and its quasi-hole are clearly allowed states in the lowest level (cf. section 3.1). The quasi-particle over the Laughlin state is obtained by having one particle in the second Landau level, leading to the form (3.24) involving one $\overline{A}$ only, i.e. $\ell = N-1, p = 0, n_N = 0$,

$$\begin{aligned} \Phi_{k,\ 1qp} &= \Phi_{k-1,\ gs}\ \Phi^{(II)}_{1,\ 1qp}\,, \\ \Phi^{(II)}_{1,\ 1qp} &= \varepsilon^{i_1\dots i_N}\ \left(\overline{A}\psi\right)_{i_N} \prod_{k=1}^{N-1} \left(\overline{B}^{k-1}\psi\right)_{i_k} . \end{aligned} \tag{3.27}$$

This is a quasi-particle in the inner part of the Laughlin fluid, it is non-degenerate and has the gap $\Delta E_{1qp} = \mathbf{B}$ (disregarding the confining potential) and $\Delta\mathcal{J} = -N$. Other quasi-particles are density rings that can be degenerate due to the free p parameter in (3.24). Multi quasi-particle states are obtained by inserting more than one $\overline{A}$ in $\Phi^{(II)}$, on different stems of the bush, according to (3.24): $\Phi_{k,\ \ell qp} = \Phi_{k-1,\ gs}\ \Phi^{(II)}_{1,\ \ell qp}$. Their energy is linear in the number of quasi-particles. We thus find that the projected $g = 0$ Maxwell-Chern-Simons matrix theory reproduces the Jain composite-fermion correspondence also for quasi-particle excitations [6], but with additional degeneracies.

Let us not proceed to find the states in the $g = 0$ theory with higher projections. In the $A^3 = 0$ theory, the $k = 1$ bushes may have two $\overline{A}$ matrices per stem at most, obeying the following rules (proofs are given in appendix A):

- If the bush has only one $\overline{A}$ per stem, i.e. for second-level fillings, the $\overline{A}$'s can stay on the stems at two values of the height, i.e. can form two bands.
- If there are stems with both one and two $\overline{A}$'s, then the $\overline{A}$'s can form two bands, with the extra condition for single-$\overline{A}$ stems that their $\overline{A}$'s should stay on the lowest band.

The first rule implies that the earlier $\nu = 2^*$ homogeneous state (3.25) becomes degenerate in the $A^3 = 0$ theory at the same density. On the other hand, the $A^3 = 0$ theory admits a maximal density state with $N/3$ gauge-invariant particles per level, that is unique due to the second rule:

$$\Phi_{1/3,\ gs} = \varepsilon^{i_1\dots i_N} \prod_{k=1}^{N/3} \left[\left(\overline{B}^{k-1}\psi\right)_{i_k} \left(\overline{A}\ \overline{B}^{k-1}\psi\right)_{i_{k+N/3}} \left(\overline{A}^2\ \overline{B}^{k-1}\psi\right)_{i_{k+2N/3}} \right] . \tag{3.28}$$

This state corresponds to filling fraction $\nu^* = 3$. Next, the product states,

$$\Phi_{k+1/3,\ gs} = \Phi_{k-1,\ gs}\ \Phi_{1/3,\ gs}\,, \tag{3.29}$$

obeys the $A^3 = 0$ condition for $k > 1$: these ground states realize the Jain composite-fermion construction for $\nu^* = 3$ and have the expected filling fraction $\nu = m/(mk+1)$ for $m = 3$.

The pattern repeats itself in the $A^4 = 0$ theory (cf. appendix A): there are three $\overline{A}$'s per stem at most, that can form up to three bands; however, if single and/or double-$\overline{A}$

stems are present together with the three-$\overline{A}$ stems, the $\overline{A}$'s of the former stems should stay on the lowest bands. Therefore, the maximal density state is again unique, having form analogous to (3.28) and filling $\nu^* = 4$.

In conclusion, the $A^m = 0$ projected theory possesses the following non-degenerate ground states with Jain fillings $\nu = m/(mk+1)$:

$$\Phi_{k+1/m,\ gs} \;=\; \Phi_{k-1,\ gs}\, \Phi_{1/m,\ gs}\,, \tag{3.30}$$

where

$$\Phi_{1/m,\ gs} = \varepsilon^{i_1\ldots i_N} \prod_{k=1}^{N/m} \left[\left(\overline{B}^{k-1}\psi\right)_{i_k} \; \left(\overline{A}\,\overline{B}^{k-1}\psi\right)_{i_{k+N/m}} \cdots \left(\overline{A}^m\,\overline{B}^{k-1}\psi\right)_{i_{k+(m-1)N/m}} \right] . \tag{3.31}$$

In the $A^m = 0$ theory, the lower density states that were non-degenerate in the $A^k = 0$ theories, $k < m$, become degenerate. Nevertheless, there are non-degenerate quasi-particles of the $(m-1)$ Jain state just below.

In conclusion, we have found that the ground states with homogeneous fillings of the properly projected Maxwell-Chern-Simons matrix theory reproduce the Jain pattern of the composite fermion transformation. These matrix states are unique solutions for certain (maximal) values of the density, while Jain states are judiciously chosen ansatzs among many possible multi-particle states of the ordinary Landau levels.

These results indicate that the Jain composite-fermion excitations have some relations with the D0-brane degrees of freedom and their underlying gauge invariance. Both of them have been described as dipoles. According to Jain [6] and Haldane-Pasquier [33], the composite fermion can be considered as the bound state of an electron and a hole (a vortex of the electron fluid): the reduced effective charge would then account for the smaller effective magnetic field $\mathbf{B}^*$ (3.18) felt by these excitations. On the other side, matrix gauge theories, such as the Maxwell-Chern-Simons theory, are equivalent to noncommutative theories whose fundamental degrees of freedom are dipoles. Clearly, a better understanding of the potential term $\mathrm{Tr}[X,\overline{X}]^2$ in our matrix theory is necessary to clarify the dipole description.

We finally remark that the matrix coordinates are less noncommutative on the Jain states then on the Laughlin ones. Indeed, the general form of the Gauss law (2.4) can be rewritten in terms of $X, \overline{X}, A, \overline{A}$ as follows:

$$[X,\overline{X}] \;+\; \frac{2}{B}\,[\overline{X},A] \;+\; \frac{2}{B}\,[\overline{A},X] \;=\; 2\left(\theta - \frac{1}{B}\psi\otimes\overline{\psi}\right) . \tag{3.32}$$

On the Laughlin states belonging to the lowest Landau level, this reduces to the coordinates noncommutativity (2.17), because $A = \overline{A} = 0$; on states populating higher levels, there are other terms contributing to noncommutativity besides the matrix coordinates. In section four, we shall discuss the theory in the opposite $g = \infty$ limit, where $[X,\overline{X}] = 0$, and thus non-commutativity is entirely realized between coordinates and momenta.

3.5 Generalized Jain's hierarchical states

In the $A^m = 0$ projected theories with $m \geq 3$, there are other solutions of the Gauss law for $k > 1$ besides the Jain states (3.30). Any combination of the $k = 1$ solutions (3.31) is possible, as follows:

$$\Phi_{\frac{1}{p_1}+\cdots+\frac{1}{p_k},\ gs} = \prod_{i=1}^{k} \Phi_{\frac{1}{p_i},\ gs} ,$$
$$\frac{1}{\nu} = 1 + \sum_{i=1}^{k} \frac{1}{p_i} . \tag{3.33}$$

In this equation, we also wrote the associated filling fractions using eq. (3.7), i.e. assuming homogeneous densities. The states (3.33) obey the condition $A^q = 0$ with $q = 1+\sum_{i=1}^{k}(p_i - 1)$. The Jain mapping to a single set of $\nu^* = q$ effective Landau levels does not hold for these generalized states. Actually, analogous states were considered by Jain as well [6], and disregarded as unlikely further iterations of the composite-fermion transformation. In the matrix theory, we seek for arguments to disregard them as well.

Let us compare the generalized (3.33) and standard (3.30) Jain states at fixed values of the background k (keeping in mind that the physical values are $k = 2, 4$). The energy of the generalized states is additive in the $\nu^* = p_i$, $k = 1$, blocks and reads:

$$E_{\frac{1}{p_1}+\cdots+\frac{1}{p_k},\ gs} = \frac{\mathbf{B}N}{2} \sum_{i=1}^{k} (p_i - 1) + V_C . \tag{3.34}$$

The analysis of some examples of fillings and energies makes it clear that these additional solutions have in general higher energies for the same filling or are more compact for the same energy than the standard Jain states (3.30) (see table 1). States of higher energies are clearly irrelevant at low temperatures. Furthermore, higher-density states strongly deviate from the semiclassical incompressible fluid value $\nu = 1/(k + 1)$ for background $\mathbf{B}\theta = k$, that is specific of the Laughlin factors [13]. This fact indicates that they might not be incompressible fluids with uniform densities. Further discussion of this point is postponed to section four.

4. $g \to \infty$ limit and electron theory

In this section we switch on the potential $V = -(g/2)\mathrm{Tr}[X_1, X_2]^2$ in the Hamiltonian (2.3) and perform the $g \to \infty$ limit. The potential is a quartic interaction between the matrices that does not commute with the Landau term, $\mathbf{B}\mathrm{Tr}\left(A^\dagger A\right)$: thus, the $g = 0$ eigenstates obtained in the previous section by filling a given number of Landau levels will evolve for $g > 0$ into mixtures of states.

At the classical level, the V potential suppresses the matrix degrees of freedom different from the eigenvalues, and projects them out for $g \to \infty$. This can be seen by using the Ginibre decomposition of complex matrices [34], which reads: $X = \overline{U}(\Lambda + R)U$, where U is unitary (the gauge degrees of freedom), Λ diagonal (the eigenvalues) and R complex upper

m	$\#p_i = 1$	$\# p_i = 2$	$\#p_i = 3$	$\#p_i = 4$	$1/\nu$	$E/\mathbf{B}$
1	k				k+1	0
2	k-1	1			k+ 1/2	N/2
3	k-2	2			k	N
3	k-1	0	1		k+1/3	N
4	k-3	3			k-1/2	3N/2
4	k-2	1	1		k-1/6	3N/2
4	k-1	0	0	1	k+1/4	3N/2

Table 1: Examples of generalized (3.33) and standard (3.30) Jain states for fixed value of k, ordered by Landau level m with corresponding fillings ν and energies E (disregarding the confining potential). Note that the experimentally relevant values are $k = 2, 4$ [6].

triangular (the additional d.o.f.). Inserting this decomposition in the potential, we find for $N = 2$:

$$V = \frac{g}{8}\mathrm{Tr}\left[X, \overline{X}\right]^2 = \frac{g}{4}|r|^4 + \frac{g}{4}|\overline{r}\,(\lambda_1 - \lambda_2)\,|^2\,, \qquad X = \begin{pmatrix} \lambda_1 & r \\ 0 & \lambda_2 \end{pmatrix} . \tag{4.1}$$

Thus for large g, the variable r is suppressed. For general N, the potential kills all the $N(N-1)$ real degrees of freedom contained in the R matrix.

Let us now discuss the matrix theory in the $g = \infty$ limit, i.e. for $R = 0$: X and $\overline{X}$ commute among themselves (they are called "normal" matrices [26]) and can be diagonalized by the same unitary transformation:

$$X = \overline{U}\Lambda U, \qquad \overline{X} = \overline{U}\overline{\Lambda}U, \qquad \Lambda = diag\ (\lambda_a)\,, \qquad \left[X, \overline{X}\right] = 0. \tag{4.2}$$

In the $g = \infty$ limit, we analyze the theory following a different strategy from that of section 3: we fix the gauge invariance, solve the Gauss law at the classical level and then quantize the remaining variables, which are the complex eigenvalues λ_a and their conjugate momenta p_a, following the analysis of refs. [27, 16]. We take the diagonal gauge for the matrix coordinates and decompose the momenta $\Pi, \overline{\Pi}$, in diagonal and off-diagonal matrices, respectively called p and Γ:

$$X = \Lambda\,, \qquad \Pi = p + \Gamma\,, \quad \overline{\Pi} = \overline{p} + \overline{\Gamma}\ . \tag{4.3}$$

The Gauss law constraint (2.4) can be rewritten:

$$\begin{aligned} [X, \Pi] + \left[\overline{X}, \overline{\Pi}\right] \ &= -i\ \mathbf{B}\theta + i\ \psi \otimes \overline{\psi}\,, \\ (\lambda_a - \lambda_b)\,\Gamma_{ab} + \left(\overline{\lambda}_a - \overline{\lambda}_b\right)\overline{\Gamma}_{ab} \ &= -i\left(k\ \delta_{ab} - \psi_a\ \overline{\psi}_b\right)\ . \end{aligned} \tag{4.4}$$

The second of (4.4) implies $|\psi_a|^2 = k$ for any value of $a = b$. We can further fix the remaining $\mathrm{U}(1)^N$ gauge freedom by choosing $\psi_a = \sqrt{k}$, $\forall a$, such that the r.h.s. of eq. (4.4) becomes proportional to $(1 - \delta_{ab})$.

Therefore the Gauss law completely determines the off-diagonal momenta: their rotation invariant form is,

$$\Gamma_{ab} \ = \ \frac{ik}{2}\,\frac{\overline{\lambda}_a - \overline{\lambda}_b}{|\lambda_a - \lambda_b|^2}\,, \qquad a \neq b\ . \tag{4.5}$$

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By inserting this back into the Hamiltonian (2.3), we find that diagonal and off-diagonal terms decouple and we obtain,

$$
\begin{aligned}
H &= 2\ \mathrm{Tr}\left[\left(\frac{\overline{X}}{2} - i\ \Pi\right)\left(\frac{X}{2} + i\ \overline{\Pi}\right)\right] \\
&= 2\sum_{a=1}^{N}\left(\frac{\overline{\lambda}_a}{2} - ip_a\right)\left(\frac{\lambda_a}{2} + i\overline{p}_a\right) + \frac{k^2}{2}\sum_{a\neq b=1}^{N}\frac{1}{|\lambda_a - \lambda_b|^2}\ .
\end{aligned}
\tag{4.6}
$$

The same result is obtained starting from the Lagrangian (2.1) and solving for A_0 in the gauge $X = \Lambda$ at $g = \infty$ [16].

Therefore, the theory reduced to the eigenvalues corresponds to the ordinary Landau problem for N electrons plus an induced two-dimensional Calogero interaction. Note also that the matrix measure of integration (3.3) reduces to the ordinary expression after incorporating one Vandermonde factor $\Delta(\lambda)$ in the wave functions [26]. The occurrence of the Calogero interaction is a rather common feature of matrix theories reduced to eigenvalues: the induced interaction is analog to the centrifugal potential appearing in the radial Schroedinger equation. In the present case, the interaction is two-dimensional, owing to the presence of two Hermitean matrices, and thus it is rather different from the exactly solvable one-dimensional case [14, 32].

We conclude that the Maxwell-Chern-Simons matrix theory in the $g = \infty$ limit makes contact with the physical problem of the fractional quantum Hall effect: the only difference is that the Coulomb repulsion e^2/r is replaced by the Calogero interaction k^2/r^2. Numerical results [5, 28, 6, 29] indicate that quantum Hall incompressible fluid states are rather independent of the detailed form of the repulsive potential at short distance, for large $\mathbf{B}$. In particular, the Calogero potential does not have the long-range tail of the Coulomb interaction and is closer to the class of much-used Haldane short-range potentials [28]. Although the physics of incompressible fluids is universal, the form of the potential might affect the detailed quantitative predictions of the theory for some quantities such as the gap: this issue is postponed to the future.

Some remarks are in order:

- The physical condition imposed by the Gauss law (4.4) is still that outlined in section 3.2: it forces the electrons to stay apart by locking their density to the value of the background parameter k. The solution of this constraint is however rather different at the two points $g = 0$ and $g = \infty$: for $g = 0$, it is the geometric, or kinematic, condition of noncommutativity (2.17), while at $g = \infty$ this is a dynamical condition set by a repulsive potential with appropriate strength.

- Such dynamical condition is far more complicate to solve, and it allows many more excited states than the kinematic condition; there are many more available states in the lowest Landau level at $g = \infty$ than in the $g = 0$ matrix theory.

- Note also that the $g = \infty$ theory is not, by itself, less difficult than the ab-initio quantum Hall problem: the gap is non-perturbative and there are no small parameters.

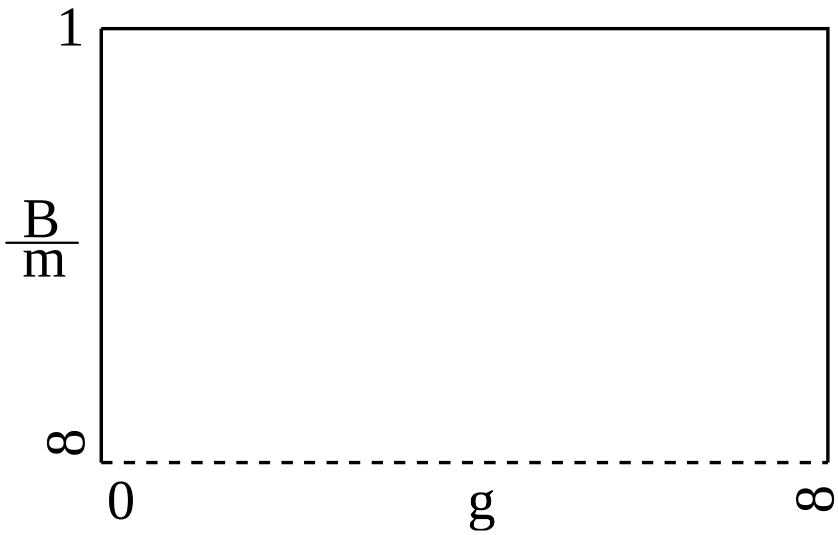

Figure 3: Phase diagram of the Maxwell-Chern-Simons matrix theory. The axes $g = 0$ and $g = \infty$ have been discussed in sections 3 and 4, respectively. The Chern-Simons matrix model sits at the left down corner.

The advantage of embedding the problem into the matrix theory is that of making contact with the solvable $g = 0$ limit, as discussed in the next section.

5. Conjecture on the phase diagram and conclusions

In figure 3 we illustrate the phase diagram of the Maxwell-Chern-Simons matrix theory as a function of its parameters $\mathbf{B}/m$ and g. The quantized background charge $\mathbf{B}\theta = k$ is held fixed over the diagram together with the parameters ω, ω_n in the confining potential (3.21).

The axes $g = 0$ and $g = \infty$ have been discussed in sections 3 and 4, respectively. For $g = 0$, the theory is solvable and displays a set of states that are in one-to-one relation with the Laughlin and Jain ground states with filling fractions $\nu = m/(mk + 1)$. These non-degenerate states can be selected by choosing the appropriate projection $A^m = 0$ and the value of k, and by tuning ω, ω_n. For $g = \infty$, we found that the theory describes the real fractional Hall effect, but we do not know how to solve the Calogero interaction and find the ground states.

Let us consider the evolution of one Jain state as g is switched on, while keeping the other parameters fixed. Given that the potential $\mathrm{Tr}\left[X, \overline{X}\right]^2$ does not commute with the $g = 0$ Landau Hamiltonian, this state will mix with other ones. If it remains non-degenerate as g grows up to infinity, we can say that the matrix theory remains in the same universality class and that the qualitative features found at $g = 0$ remain valid in the physical limit $g = \infty$. In the case of level crossing at some finite value $g = g^*$, the two regimes of the theory are unrelated.

Unfortunately, we do not presently have a method of solution of the $g \neq 0$ Hamiltonian, even approximate, that could prove the non-degeneracy of Jain states for $g > 0$ and establish the physical relevance of the Maxwell-Chern-Simons matrix theory.

Nevertheless, we would like to conjecture that the Laughlin and Jain states at $g = 0$ do remain non-degenerate. Namely, that there is no phase transition at finite g values when the theory is tuned on such ground states at $g \sim 0$ (by appropriate choices of m, k, ω, ω_n).

Our conjecture is indirectly supported by the numerical results by Jain and others [28, 6, 29], through the following classical argument. These authors found that the Laughlin and Jain states in the quantum Hall effect are very close to the exact numerical ground states for a variety of short-range potentials, including the Calogero one realized at $g = \infty$. Now, consider the $g > 0$ evolution of the Jain matrix ground states: the effect of the potential can be seen, at the classical level, as that of eliminating the additional matrix d.o.f. and make both $X, \overline{X}$ matrices diagonal (up to a gauge transformation, cf. section 4). In this case, the Jain matrix states become Slater determinants of Hall states (cf. section 3.4) and exactly reduce to the expressions introduced by Jain [6]. Therefore, it is rather reasonable to expect that the evolution the $g = 0$ matrix states will bring them into the diagonalized, i.e. original Jain states at $g = \infty$, up to small deformations.

On the contrary, other states such as those of the generalized Jain hierarchy (cf. section 3.5), that have no counterpart in the $g = \infty$ theory, are likely to become degenerate at finite g.

In conclusion, our conjecture of smooth evolution of matrix Jain states is supported by the numerical analyses of the Jain composite-fermion theory. Let us add some comments:

- The projected $A^m = 0$ theory describes the $\nu^* = m$ Jain matrix state and its quasi-hole excitations that are gapful at $g = 0$: our conjecture is that this gap does not vanish for $g > 0$. On the other hand, the quasi-particle gap cannot be discussed in the $A^m = 0$ theory, because the quasi-particles live in the theory with higher projection, A^{m+1}, where the $\nu^* = m$ state is itself degenerate. The projections complicate the $g = 0 \leftrightarrow \infty$ correspondence and might be relaxed at some point of the g-evolution; this remains to be understood.

- As discussed in section 3, the limit $\mathbf{B} \to \infty$ cannot be taken at $g = 0$, because quasi-particle excitations and Jain states in the matrix theory have energies of $O(\mathbf{B})$ and would be projected out. Instead, the limit $\mathbf{B} = \infty$ can surely be taken in the $g = \infty$ physical theory (holding $k = \mathbf{B}\theta$ fixed), because the fractional quantum Hall states are known to remain stable. This implies that the two limits are ordered: the correct sequence is $\lim_{\mathbf{B}\to\infty} \lim_{g\to\infty} \Psi$, and the opposite choice is cut out in the phase diagram of figure 3.

- The projection to the lowest Landau level, $\mathbf{B} \to \infty$ at $g \gg 1$ should also transform the Jain states into incompressible fluids with constant densities corresponding to the filling fractions assigned before. It is rather clear that the Jain matrix states at $g = 0$ (section 3.4) are not uniform: they are multiple-droplet states similar to those considered in [15]. On the other hand, the matrix Laughlin states (3.11)

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do correspond to incompressible fluids in the lowest Landau level as shown in the refs. [13, 14].

In summary, in this paper we have generalized the Susskind-Polychronakos proposal of noncommutative Chern-Simons theory and matrix models. We have found:

- A description of the expected Jain states and their quasi-particle excitations within a matrix generalization of the Landau levels.
- An interesting phase diagram, parametrized by the additional coupling g, with a manifestly physical limit for the matrix theory at $g = \infty$.

Reliable methods of solution for the potential g Tr $\left[X, \overline{X}\right]^2$ are needed to understand the phase diagram, verify the proposed physical picture and allow for the physical applications. Recent developments in the analysis of multi-matrix theories [35] may provide new tools for tackling this problem.

Acknowledgments

We warmly thank A. Polychronakos for many discussions and suggestions on this work. We also thank M. Riccardi, D. Seminara and G. R. Zemba for interesting discussions. A. Cappelli thanks the KITP, Santa Barbara, for hospitality. This research was supported in part by the National Science Foundation under Grant No. PHY99-07949. I. D. Rodriguez thanks the EC program Alban of Ph-D scholarships for Latin American students. This work was partially funded by the EC Network contract HPRN-CT-2002-00325, "EUCLID: Integrable models and applications: from strings to condensed matter", and by the ESF programme "INSTANS: Interdisciplinary Statistical and Field Theory Approaches to Nanophysics and Low Dimensional Systems".

A. Projections of matrix Landau states

In this appendix we prove some properties of the projections of states described in section 3.4. Let us first show that the general solution of the second Landau level projection (3.22),

$$\left(\frac{\partial}{\partial \overline{A}_{ab}}\right)^2 \Phi(\overline{B}, \overline{A}, \psi) \;=\; 0\,, \qquad \forall\, a, b\,. \tag{A.1}$$

and of the $k = 1$ Gauss law (3.5) is given by the expression (3.24). We start from the gauge invariant expressions involving N fields ψ,

$$\Phi = \varepsilon^{a_1 \ldots a_N}\ \left(M_1\psi\right)_{a_1} \cdots \left(M_N\psi\right)_{a_N}\ , \tag{A.2}$$

where the M_i are polynomials of $\overline{B}$ and $\overline{A}$. In this appendix, we repeatedly use the graphical description of these expressions in terms of bushes as shown in figure 1. Upon expanding (A.2) into monomials, we get a sum of bushes:

$$\Phi = \varepsilon^{a_1 \ldots a_N}\left(P_1\psi\right)_{a_1} \cdots \left(P_N\psi\right)_{a_N} + b\,\varepsilon^{a_1 \ldots a_N}\left(Q_1\psi\right)_{a_1} \cdots \left(Q_N\psi\right)_{a_N} + c\,\varepsilon \ldots R_1 \cdots R_N + \cdots, \tag{A.3}$$

where the monomials in a bush, e.g. the $\{P_i\}$, are all different among themselves, and two sets of monomials, e.g. $\{P_i\}$ and $\{Q_j\}$, differ in one monomial (stem) at least.

The two derivatives in (A.1) act in all possible ways on the stems of the bushes, and can be represented by primed expressions, i.e. $P_i', P_i'', \ldots$. Let us momentarily take bushes made of two stems, i.e. $N = 2$:

$$\begin{aligned}\Phi'' &= \varepsilon_{ab}\left(P_1''\psi_a\ P_2\psi_b + 2P_1'\psi_a\ P_2'\psi_b + P_1\psi_a\ P_2''\psi_b\right) \\ &\quad + b\ \varepsilon_{ab}\left(Q_1''\psi_a\ Q_2\psi_b + 2Q_1'\psi_a\ Q_2'\psi_b + Q_1\psi_a\ Q_2''\psi_b\right) + \cdots . \end{aligned} \tag{A.4}$$

We check the possibility of cancellations between terms belonging to two different bushes: these cannot occur between terms with the same pattern of derivatives, i.e. $P_1\ P_2''$ and $Q_1\ Q_2''$, because at least one monomial is different between the two bushes: $P_1 \neq Q_1$ or $P_2 \neq Q_2$. There can be cancellations between terms that have different derivatives, i.e. $P_1''\ P_2 + b\ Q_1'\ Q_2' = 0$, but then the symmetric term would not cancel, $P_1\ P_2'' + b\ Q_1'\ Q_2' \neq 0$. We conclude that there cannot be complete cancellations between two bushes and that each bush should vanish independently.

Consider now the action of derivatives on the stems of a single bush; the terms with two derivatives, i.e $P_i''\psi$, should vanish independently, because the stems in bush are all different. Thus, there cannot be more than one $\overline{A}$ per stem. Next, we distribute one derivative per stem: each of them cuts the $\overline{A}_{ab}$ from the stem leaving fixed indices at the end points, leading to the expression,

$$\left(\frac{\partial}{\partial \overline{A}_{ab}}\right)^2 \Phi = \varepsilon_{cd}\ \overline{B}^{n_1}_{ca}\ \overline{B}^{n_2}_{da}\quad \left(\overline{B}^{m_1}\psi\right)_b \left(\overline{B}^{m_2}\psi\right)_b\ . \tag{A.5}$$

This vanishes by antisymmetry of the epsilon tensor, provided that $n_1 = n_2$, i.e. that the $\overline{A}$ matrices appear at the same level on the two stems. Furthermore, for $N > 2$ one can repeat the argument, having $N - 2$ spectator stems over which the derivatives do not act; the height condition should then applies for any pair of stems that have one $\overline{A}$. In conclusion, all $\overline{A}$ should appear in the stems at the same height, leading to the general solution (3.24).

A.1 States obeying the $A^3 = 0$ projection

We now discuss the solution of the $A^3 = 0$ condition. Bushes can have one, two, three $\overline{A}$'s per stem and more: we consider each case in turn. For three $\overline{A}$'s and more, $A^3 = 0$ can act on a single stem and not vanish: this limits the number of $\overline{A}$'s per stem to two.

(A) For bushes that have single-$\overline{A}$ stems only, we should examine the action of A^3 an all triples of stems (1-1-1 action). This vanishes by antisymmetry (cf. (A.5)) if for any triple considered, two $\overline{A}$'s are at the same height. It follows that on single-$\overline{A}$ bushes, the $\overline{A}$'s can stay at two heights, i.e. form two bands.

(B) For bushes with double-$\overline{A}$ stems only, A^3 can act on pairs (2-1 action (B_1)) or on triples (1-1-1 action (B_2)) of stems.

(B_1) Consider the action 2-1 on the pair:

$$\left(\frac{\partial}{\partial \overline{A}_{ab}}\right)^3 \varepsilon_{ij}(C\overline{A}D\overline{A}E)_i\ (F\overline{A}G\overline{A}H)_j = \varepsilon_{ij}[(C_{ia}D_{ba}E_b)(F_{ja}G\overline{A}H_b) + \\ +(C_{ia}D_{ba}E_b)\ (F\overline{A}G_{ja}H_b) + \\ +(C_{ia}D\overline{A}E_b)\ (F_{ja}G_{ba}H_b) + \\ +(C\overline{A}D_{ia}E_b)\ (F_{ja}G_{ba}H_b)]. \quad \text{(A.6)}$$

The first and third term in this equation vanish independently when $C = F$ due to the earlier identity $\varepsilon_{ij}u_iu_j = 0$; the sum of the second and fourth term vanishes for $D = G$ due to the possibility of factorizing an expression of the type, $\varepsilon_{ij}\,(u_iv_j + u_jv_i) = 0$. We thus found that the double-$\overline{A}$ stems should have $\overline{A}$ located on two heights (two bands).

(B_2) There are $2^3 = 8$ possible actions 1-1-1 on triples of stems involving two $\overline{A}$'s each. Having already enforced condition (B_1), their $\overline{A}$'s are located on two bands. The 8 terms generated by the action of A^3 are found vanish by the same two mechanism found in (A.6). Therefore, there are no new conditions.

(C) For bushes involving both double- and single-$\overline{A}$ stems, we should again consider the actions 2-1 on pairs (C_1) and 1-1-1 on triples (C_2) of stems.

(C_1) We consider the pair made by one double-$\overline{A}$ stem and one single-$\overline{A}$ stem; the double derivative acts necessarily on the former stem, thus producing a unique term. This vanishes as $\varepsilon_{ij}u_iv_j = 0$ if $u = v$, namely if the $\overline{A}$ on the single stem-$\overline{A}$ is located at the same height of the lower $\overline{A}$ in the double-$\overline{A}$ stem. It implies that the $\overline{A}$ form again two bands, but those on single-$\overline{A}$ stems should stay in the lower band.

(C_2) The three derivatives act 1-1-1 on triples of stems with number of $\overline{A}$'s equal to (2,1,1) or (2,2,1), yielding 2 and 4 terms respectively. All these terms vanish independently, because single-$\overline{A}$ stems already have their $\overline{A}$ on the lowest band by condition C_1.

In summary, the $A^3 = 0$ projection allows two $\overline{A}$ per stem at most, that should form two bands. If both single- and double-$\overline{A}$ stems are present in the same bush, the $\overline{A}$ on single stems should stay on the lower band. All these features have been checked on the computer for small-N examples.

A.2 States with $A^4 = 0$ projection

Again the action of the four derivatives on a single stem is not vanishing and requires three $\overline{A}$ per stem at most. Hereafter we list the possible actions of the four derivatives.

(A) If there are single-$\overline{A}$ stems only, the derivative action is 1-1-1-1: for every four-plet of stems, two $\overline{A}$ should be at the same height; thus, three bands of $\overline{A}$ can be formed on bushes.

(B) If there are double-$\overline{A}$ stems only, there can be: (B_1) 2-2 action on pairs of stems; (B_2) 2-1-1 action on triples of stems, (B_3) 1-1-1-1 action on four-plets of stems.

- (B_1) There is a single term that vanishes if the lower $\overline{A}$ are at the same height.
- (B_2) There are 12 terms that vanish by the same two mechanisms of B_1 in the previous $A^3 = 0$ case, provided the upper $\overline{A}$ form another band, i.e. stay at the same height.
- (B_3)All terms vanish once the previous conditions are enforced.

In summary, double-$\overline{A}$ stems should have their $\overline{A}$'s on two bands.

(C) If there are triple-$\overline{A}$ stems only, there can be: (C_1) 3-1 and 2-2 actions on pairs; (C_2) 2-1-1 action on triples; (C_3) 1-1-1-1 on four-plets.

- (C_1) There are 6 terms for the 3-1 action and 9 for the 2-2 action: these cancel individually or in pairs by the two mechanisms of B_1 in the previous $A^3 = 0$ case, provided that all $\overline{A}$'s form three bands.
- (C_2) The action 2-1-1 on 3-$\overline{A}$ stems generates 81 terms, that are satisfied once C_1 has been enforced.
- (C_3) The terms generated by the 1-1-1-1 action vanish because there are at least two derivatives of $\overline{A}$ at the same height.

In summary, triple-$\overline{A}$ stems should have their $\overline{A}$'s on three bands.

(D) Consider now the case of stems having two or one $\overline{A}$ each, as for states filling the second and third Landau level. From the previous analysis we know that the double-$\overline{A}$ form 2 bands (case (B)) and the triple-$\overline{A}$ stems can have 3 bands (case (C)). We should only consider the new cases when the four derivatives act on stems of mixed type. There can be: (D_{11}) 2-1-1 action on 2-2-1 stems; (D_{12}) 2-1-1 action on 2-1-1 stems; (D_{21}) 1-1-1-1 action on 2-2-2-1 stems; (D_{22}) 1-1-1-1 action on 2-2-1-1 stems; (D_{23}) 1-1-1-1 action on 2-1-1-1 stems;

- (D_{11}) Given that one derivative acts on the single-$\overline{A}$ stem, the remaining three derivatives cancel as in case (B_1) of $A^3 = 0$, on stems already having two $\overline{A}$ bands. No new conditions.
- (D_{12}) The condition is that on any pair of single-$\overline{A}$ stems, one of them has the $\overline{A}$ on the lowest band of the double-$\overline{A}$ stems. This allows single-$\overline{A}$ to stay on any of the two bands, with some exceptions.
- (D_{21}) It is satisfied.
- (D_{22}) It yields the same condition as (D_{12}).
- (D_{23}) For every triple of single-$\overline{A}$ stems, two should be on the same band. The solution is that each of the 2 bands of double-$\overline{A}$ stems are allowed (weaker than (D_{12})).

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In summary, mixed double- and single-$\overline{A}$ stems should have their $\overline{A}$'s forming two bands with some exceptions.

(E) The most relevant case for Jain's ground state at $\nu^* = 4$ is for mixed stems with one, two and three $\overline{A}$'s. Owing to the previous conditions, each individual type is already organized in 3, 2 and 3 bands respectively. The possible new actions of the four derivatives are the following ones: (E_{11}) 3-1 and 2-2 actions on pairs of type 3-2; (E_{12}) 3-1 action on pairs of type 3-1; (E_{21}) 2-1-1 action on triples of type 3-3-2, (E_{22}) on triples 3-2-2, (E_{23}) on triples 3-2-1 and (E_{24}) on triples 3-1-1 ; (E_4) 1-1-1-1 actions on all stem types.

(E_{11}) There is cancellation by the usual two mechanisms ((B_1) of $A^3 = 0$) provided that the $\overline{A}$'s of double-$\overline{A}$ stems stay in the lowest of the three bands of triple-$\overline{A}$ stems.

(E_{12}) As before, the $\overline{A}$ of single-$\overline{A}$ stems should all align on the lowest of the three bands of the triple-$\overline{A}$ stems.

Once these two conditions are enforced, the other E-type actions are checked. In summary, mixed triple-, double- and single-$\overline{A}$ stems should have their $\overline{A}$'s forming three bands, with the condition that stems with less that three $\overline{A}$'s should align their $\overline{A}$'s on the lowest available bands. This is the condition enforcing the uniqueness of the state with maximal filling $\nu^* = 4$ as explained in section 3.4. The same mechanism works for the $A^m = 0$ ground states with $\nu^* = m$ (3.31) that contain stems of any number of $\overline{A}$'s.

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